International Journal of Research in Advanced Engineering and Technology
www.allengineeringjournal.in
ISSN: 2455-0876
Received: 23-05-2026, Accepted: 22-06-2026, Published: 07-07-2026
Volume 12, Issue 3, 2026, Page No. 34-38

Mathematical modeling of inventory systems with fixed shelf life

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DOI: https://doi.org/10.66856/ijraet.2026.12.3.12028

Abstract
Inventory systems containing products with a fixed shelf life differ fundamentally from conventional inventory systems because physical age affects both availability and economic value. A replenishment policy that is optimal for nonperishable products may generate excessive waste when items expire before demand occurs. This paper develops a mathematical framework for inventory systems with deterministic shelf life, stochastic demand, replenishment lead time, and age-dependent holding and disposal costs. The model represents inventory as an age-structured state variable and derives the associated balance equations, demand fulfillment constraints, expiration dynamics, cost function, and long-run optimization objective. A discrete-age formulation is also developed for computational implementation. The paper further discusses first-expired-first-out issuing, lost sales, backlogging, stochastic demand, and numerical solution methods. A numerical illustration demonstrates the trade-off between product availability and expiration waste. The framework is applicable to food, pharmaceuticals, blood products, chemicals, and other products for which inventory becomes unusable after a prescribed lifetime.

Keywords: fixed shelf life, perishable inventory, age-structured model, expiration, stochastic demand, renewal policy, inventory optimization

Introduction
The management of inventory with finite lifetime is important in industries where products deteriorate or become unusable after a fixed period. Examples include vaccines, fresh food, blood components, pharmaceutical products, and temperature-sensitive materials. In such systems, inventory quantity alone is insufficient to describe the state of the system. Two inventories having the same number of units may have substantially different future value if their age distributions differ.
A fixed-shelf-life inventory system has three principal characteristics:
- Units arrive through replenishment;
- Demand removes units according to an issuing policy;
- Units that reach their maximum age expire and are discarded or salvaged.

The central difficulty is the interaction between replenishment and aging. Excessive replenishment increases service levels but also increases expiration waste. Insufficient replenishment reduces waste but may increase shortages and lost sales. Therefore, the optimal policy must balance procurement, holding, shortage, and disposal costs.
Conventional models based only on total stock cannot fully capture this interaction. An age-structured formulation is required. This paper develops such a formulation and presents a tractable discrete-age model for inventory control.

Literature Background
Perishable inventory theory has developed through several related modeling traditions. Early studies considered inventory systems in which products deteriorate continuously or expire after a fixed lifetime. Later work incorporated stochastic demand, replenishment lead times, lost sales, backlogging, salvage values, and service-level constraints.

The main modeling approaches are:
1. **Fixed-lifetime models**, in which each unit remains usable for exactly L time units.
2. **Random-lifetime models**, in which product lifetime follows a probability distribution.
3. **Continuous-deterioration models**, in which usable inventory decreases gradually.
4. **Age-structured models**, in which inventory is represented by the quantity in each age class.
5. **Fluid and diffusion approximations**, which replace discrete demand and replenishment events with continuous approximations.

The fixed-lifetime case is particularly useful for products regulated by expiration dates. It also provides a clear benchmark for evaluating more general deterioration models.
The model developed below is related to the classical work on perishable inventory systems by Nahmias, and to age-based issuing policies such as first-expired-first-out. The emphasis here is the integration of an age-structured state description with a total-cost optimization objective.

Model Assumptions and Notation
Consider a single-item inventory system operating over a planning horizon $[0, T]$.
3.1 Assumptions
The following assumptions are adopted:
- Every replenished unit has a deterministic shelf life of L time units.
- Demand occurs according to a stochastic counting process.
- Replenishment quantities are nonnegative.
- Replenishment may be subject to a deterministic or stochastic lead time.

- Products are issued according to the first-expired-first-out policy.
- Expired units cannot satisfy demand.
- Unsatisfied demand is either lost or backlogged, depending on the selected model.
- Holding, shortage, disposal, and ordering costs are included.
- The initial age distribution is known.

The fixed-shelf-life assumption implies that a unit received at time u is usable during the interval $[u, u+L)$ and expires at time $u+L$.

3.2 Continuous age representation

Let $x(a,t)$ denote the quantity of usable inventory having age a at time t, where

$$0 \le a < L.$$

The total usable inventory is

$$I(t) = \int_0^L x(a,t)\,da \qquad (1)$$

As time increases, every surviving unit becomes older. In the absence of replenishment and demand, the age density satisfies the transport equation

$$\frac{\partial x(a,t)}{\partial t} + \frac{\partial x(a,t)}{\partial a} = 0, \qquad 0 < a < L \qquad (2)$$

This equation expresses the fact that age increases at unit rate.
Demand removes inventory from the youngest or oldest available age class depending on the issuing policy. Under first-expired-first-out, demand is allocated to the oldest available units first. Let $\delta(a,t)$ denote the rate at which demand removes inventory of age a. The general balance equation becomes

$$\frac{\partial x(a,t)}{\partial t} + \frac{\partial x(a,t)}{\partial a} = -\delta(a,t), \qquad 0 < a < L \qquad (3)$$

The demand-removal function must satisfy

$$\int_0^L \delta(a,t)\,da = \min\{d(t), I(t)\} \qquad (4)$$

Where $d(t)$ is the instantaneous demand rate.
Under first-expired-first-out, there exists an age threshold $a^*(t)$ such that inventory is removed from ages greater than or equal to $a^*(t)$ before younger inventory is used.

3.3 Replenishment boundary condition

Let $r(t)$ denote the replenishment arrival rate. Newly received units have age zero. Therefore, the boundary condition at age zero is

$$x(0,t) = r(t) \qquad (5)$$

For an order placed at time τ_k with quantity q_k and lead time ℓ_k, the arrival occurs at

$$t_k^{\text{arr}} = \tau_k + \ell_k \qquad (6)$$

If replenishment is represented as an impulse, then

$$r(t) = \sum_{k=1}^{K} q_k\,\delta(t - \tau_k - \ell_k) \qquad (7)$$

where $\delta(t - \tau_k - \ell_k)$ is the Dirac delta function.

3.4 Expiration flow

Units expire when their age reaches L. The expiration flow is the outgoing boundary flux

$$E(t) = x(L^-, t) \qquad (8)$$

where L^- denotes the age immediately before expiration.
The total quantity expired over the horizon is

$$E_T = \int_0^T E(t)\,dt \qquad (9)$$

If expiration generates a disposal cost, the total disposal cost is

$$C_e = c_e E_T \qquad (10)$$

If expired products have salvage value v_s, then the corresponding net expiration cost is

$$C_e = (c_e - v_s)E_T \qquad (11)$$

Demand and Shortage Dynamics

Let $D(t)$ be the cumulative demand process. The demand increment over a small interval $[t, t+\Delta t)$ A common assumption is that demand follows a Poisson process with rate λ. In that case,

$$\Pr\{\Delta D(t) = m\} = \frac{(\lambda \Delta t)^m \exp(-\lambda \Delta t)}{m!}, \qquad m = 0,1,2,\ldots \qquad (12)$$

For a lost-sales system, the fulfilled demand is

$$F(t) = \min\{D(t), I(t)\}$$

and the lost-sales quantity over the horizon is

$$W_T = D(T) - F(T)$$

For a backlogging system, let $B(t)$ denote the backlog. The net inventory position is

$$Z(t) = I(t) - B(t)$$

The backlog dynamics are

$$\frac{dB(t)}{dt} = d(t) - f(t) \qquad (13)$$

where $f(t)$ is the rate of fulfilled demand. The constraints are

$$0 \le f(t) \le d(t), \qquad f(t) \le I(t)$$

The shortage cost for a lost-sales system is

$$C_w = c_w W_T \qquad (14)$$

For a backlogging system, a time-dependent shortage cost can be written as

$$C_s = c_s \int_0^T B(t)\, dt \quad (15)$$

Discrete-Age Formulation

The continuous model is useful analytically, but a discrete-age model is often more convenient for simulation and optimization.

Divide the shelf life into M age classes of width

$$\Delta a = \frac{L}{M} \quad (16)$$

Let $I_j(t)$ denote the quantity in age class j, where, $j = 1,2,\dots,M$

Class 1 contains the newest inventory, and class M contains inventory closest to expiration. The total usable inventory is

$$I(t) = \sum_{j=1}^{M} I_j(t) \quad (17)$$

During each time interval Δt, inventory ages according to

$$I_{j+1}(t+\Delta t) = I_j(t) - u_j(t), \qquad j = 1,2,\dots,M-1 \quad (18)$$

where $u_j(t)$ is the quantity issued from class j.

The quantity expiring during the interval is

$$E(t+\Delta t) = I_M(t) - u_M(t) \quad (19)$$

The new replenishment enters the youngest class:

$$I_1(t+\Delta t) = r(t+\Delta t) \quad (20)$$

Under first-expired-first-out, demand is allocated recursively:

$$u_M(t) = \min\{I_M(t), D(t)\} \quad (21)$$

and for $j = M-1, M-2, \dots, 1$,

$$u_j(t) = \min\left\{I_j(t), \left[D(t) - \sum_{k=j+1}^{M} u_k(t)\right]^+\right\} \quad (22)$$

where

$$[z]^+ = \max\{z, 0\}.$$

The lost sales during the period are

$$W(t) = \left[D(t) - \sum_{j=1}^{M} u_j(t)\right]^+ \quad (23)$$

This discrete formulation preserves the principal feature of fixed shelf life: every unit moves deterministically from a younger to an older age class before expiring.

Cost Function and Optimization Problem

The total expected cost over the planning horizon is composed of ordering, holding, shortage, expiration, and procurement costs.

The ordering cost is

$$C_o = \sum_{k=1}^{K} \left(K_f + c_o q_k\right) \quad (24)$$

where K_f is a fixed ordering cost.

The holding cost is

$$C_h = c_h \int_0^T I(t)\, dt \quad (25)$$

For a lost-sales system, the shortage cost is $C_w = c_w W_T$

The expiration cost is $C_e = c_e E_T$

Therefore, the total cost is

$$C_T = C_o + C_h + C_w + C_e \quad (26)$$

The optimization objective is

$$\min_{\pi \in \Pi} \mathbb{E}_\pi [C_T] \quad (27)$$

Where π denotes a replenishment policy and Π is the set of feasible policies.

Equivalently,

$$\min_{\pi \in \Pi} \mathbb{E}_\pi \left[\sum_{k=1}^{K} \left(K_f + c_o q_k\right) + c_h \int_0^T I(t)\, dt + c_w W_T + c_e E_T\right] \quad (28)$$

Together with the inventory transition equations and service-level requirements.

A fill-rate constraint may be imposed as

$$\frac{\mathbb{E}[F(T)]}{\mathbb{E}[D(T)]} \geq \beta \quad (29)$$

where β is the required service level.

Renewal-Based Approximation

For a stationary replenishment policy, long-run average cost can be analyzed using renewal cycles. Let τ denote the cycle length and let $C(\tau)$ be the total cost incurred during one cycle. The long-run average cost is

$$g(\pi) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi [C_T] \quad (30)$$

Under regenerative conditions, the renewal-reward theorem gives

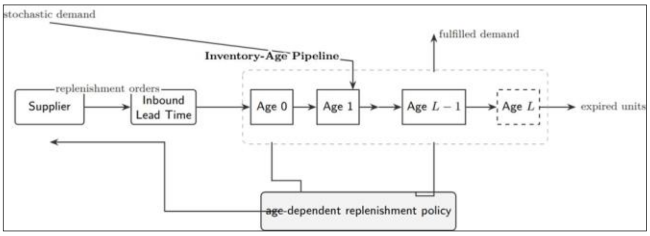
$$g(\pi) = \frac{\mathbb{E}_\pi [C(\tau)]}{\mathbb{E}_\pi [\tau]} \quad (31)$$

The optimization problem becomes

$$\min_{\pi \in \Pi} \frac{\mathbb{E}_\pi [C_o(\tau) + C_h(\tau) + C_w(\tau) + C_e(\tau)]}{\mathbb{E}_\pi [\tau]} \quad (32)$$

This representation is useful when inventory cycles begin after each replenishment or after the inventory process returns to a specified state.

Diagram



The system logic can be summarized as

$$\text{orders} \rightarrow \text{lead time} \rightarrow \text{age-structured inventory} \rightarrow \begin{cases}\text{fulfilled demand,} \\ \text{expired units.}\end{cases}$$

The policy controller observes inventory by age, demand, outstanding orders, and expiration risk.

Sensitivity Analysis

9.1 Shelf life

A shorter shelf life increases the probability of expiration before demand occurs. Thus, for otherwise fixed parameters,

$$L_1 < L_2 \quad \Rightarrow \quad E_T(L_1) \geq E_T(L_2) \tag{33}$$

under comparable replenishment policies.
The marginal value of increasing shelf life can be defined as

$$V_L = -\frac{\partial g^*}{\partial L} \tag{34}$$

where g^* is the optimized long-run average cost. A large value of V_L indicates that improved preservation, packaging, or temperature control may have substantial economic value.

9.2 Demand variability

Greater demand variability generally increases either shortage or expiration risk. If average demand remains fixed but variance increases, the safety stock required to maintain a given fill rate tends to increase. For a fixed shelf life, this additional stock may become obsolete before demand occurs.

9.3 Lead time

Longer lead time increases the inventory required to protect against stockouts. However, inventory ordered earlier may expire before replenishment is needed. Therefore, lead time and shelf life interact through the dimensionless ratio

$$\rho_L = \frac{\ell}{L} \tag{35}$$

When ρ_L is large, a substantial fraction of the product's useful lifetime may be consumed before the product becomes available for demand.

9.4 Disposal cost

An increase in expiration cost c_e shifts the optimal policy toward lower stock levels or more frequent smaller replenishments. The sensitivity of the optimized cost to disposal cost is under standard regularity conditions.

$$\frac{\partial g^*}{\partial c_e} = \text{long-run expiration rate}$$

Managerial and Operational Implications

The model leads to several practical conclusions:

- Inventory should be monitored by age, not only by total quantity.
- First-expired-first-out issuing reduces avoidable waste.
- Smaller and more frequent orders may outperform large replenishments when disposal costs are high.
- Longer shelf life can be economically valuable even when the unit purchase price is unchanged.
- Service-level targets should be evaluated jointly with expiration ratios.
- Replenishment decisions should consider outstanding orders and the remaining shelf life of current inventory.
- A high nominal inventory level does not necessarily imply high effective availability.

An operational dashboard should therefore track at least:

$$\{I(t),\ I_{\text{near-expiry}}(t),\ E(t),\ W(t),\ \text{Fill Rate},\ \overline{C}\} \tag{36}$$

Limitations

The deterministic shelf-life assumption is appropriate for products with clearly specified expiration dates, but it may not represent gradual quality loss. Real systems may also exhibit:

- Uncertain replenishment quantities;
- Temperature excursions;
- Partial demand fulfillment;
- Product substitution;
- Batch-dependent shelf lives;
- Inspection and quality-control delays;
- Capacity constraints;
- Correlated demand and supply disruptions.

Consequently, the model should be calibrated against operational data before being used for policy implementation. Simulation experiments are particularly important when demand, lead times, and product lifetimes are simultaneously stochastic.

Conclusion

A fixed-shelf-life inventory system is naturally modeled as an age-structured stochastic process. The essential state variable is not merely total inventory but the quantity of inventory at each age. The resulting model captures replenishment, demand fulfillment, aging, expiration, shortage, and cost accumulation within a unified framework. The continuous transport equation

$$\frac{\partial x}{\partial t} + \frac{\partial x}{\partial a} = -\delta$$

describes the movement and depletion of inventory through age space, while the discrete-age formulation provides a practical basis for simulation and optimization. The resulting objective balances procurement, holding, shortage, and disposal costs:

$$\min_{\pi \in \Pi} \mathbb{E}_\pi [C_o + C_h + C_w + C_e].$$

The principal research implication is that optimal replenishment should be age-sensitive. Policies based only on aggregate inventory can overstock products that are close to expiration while simultaneously underestimating future shortage risk. Integrating age information, demand uncertainty, lead time, and expiration cost provides a stronger foundation for inventory decisions in perishable-product supply chains.

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